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LETTER TO THE EDITOR

Conductance and resistance jumps in finite-size random resistor networks

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Abstract. When bonds are removed one by one and at random from a finite-size resistor network, the conductance (or resistance) does not change continuously, but rather a sequence of conductance (resistance) jumps of various sizes occurs. The larger jumps arise from those bonds which carry a relatively large current just before they are cut. We report on numerical simulations of these jumps for a random resistor network on the square lattice. We also give a scaling argument to account for this phenomenon, which yields the number of conductance jumps larger than ΔG scaling as $(\Delta G)^{-\lambda_G}$, with $\lambda_G = d\nu/(d\nu - t)$, where t is the conductivity exponent and ν is the percolation correlation-length exponent. Equivalently, the number of resistance jumps greater than ΔR scales as $(\Delta R)^{-\lambda_R}$, with $\lambda_R = d\nu/(d\nu + t)$. These predictions account for our data on a qualitative level only, however, and we discuss some possible mechanisms for the quantitative discrepancies.

In the study of random systems, one is generally interested in the behaviour of the thermodynamic limit, where physical observables are continuous functions of external parameters. For example, the conductance, G, of a random resistor network is a smooth function of the bond concentration p, and as p approaches the percolation threshold p_c from above, G vanishes as $(p - p_c)^t$, where t is the conductivity exponent [1]. However, in a finite-size realisation of the network, the behaviour is considerably different. The conductance is no longer a smooth function [2-5] because the removal of a single bond causes a finite change in the bond concentration. This effect becomes relatively more important near p_c, as a bond may be singly connected [6] on a length scale comparable to the system size. Consequently, such a bond can carry a non-trivial fraction of the total current flowing through the network. The deletion of this type of bond can then lead to a macroscopic change in the distribution of currents in the network. Furthermore, the deletion of this bond can also lead to the fact that additional bonds are no longer part of the conducting backbone. These fluctuation effects are fundamental for an understanding of current flow in random media at the microscopic level. Our goal, in this letter, is to understand these effects as embodied by the behaviour of conductance and resistance jumps as a random resistor network is depleted at random. Very similar work on this issue has been carried out simultaneously and independently by Roux and Wilkinson [7].

For the purposes of a qualitative understanding of the conductance jumps, consider first what happens when a single bond is removed from a d-dimensional lattice of linear size L, for the case where p is nearly equal to 1. In this regime, effective medium theory [1] offers an excellent approximation for the behaviour of the conductance and

from this approach, it immediately follows that $\Delta G \sim \Delta p$, where $\Delta p \sim L^{-d}$. That is, the conductance jumps are well characterised, and this is what is observed in numerical simulations and experiments on random resistor networks [2-5]. However, as $p \to p_c$, the conductance jumps are no longer so predictable, as indicated in figure 1. Most of the time, the bond just removed was part of a finite cluster or a dangling end, which consequently leads to $\Delta G = 0$. However, as discussed above, if the bond to be removed carries a substantial current, then its removal will lead to a relatively large jump in the conductance of the system.

Although the presence of large conductance jumps would seem to obviate a conventional analysis for critical exponents, their existence can nevertheless be usefully exploited for estimating exponent values. For example, in an experimental measurement of the concentration dependence of the conductance in a random resistor-diode network [5], it was observed that the size of the conductance jumps was strongly correlated with corresponding jumps in the number of bonds contained in the conducting backbone. Thus by studying the dependence of the conductance with respect to the backbone fraction, it is possible to obtain useful information about the conductivity exponent, even though the raw data is sufficiently 'jumpy' that it appears to preclude the possibility of estimating exponent values. Very recently, Thompson *et al* [8] have performed mercury injection experiments in porous media in order to probe this jump phenomenon in a direct manner. Once the 'breakthrough' of mercury has occurred (so that the system is above the percolation threshold) then, as the pressure of the injected mercury is increased further, a sequence of resistance jumps which appear to have a power law distribution of sizes is observed. These jumps are postulated to originate from the formation of additional conducting paths of mercury as the pressure is increased. On the basis of these measurements, Thompson *et al* suggest that resistance jumps might help provide information about the structure of randomly porous media at the pore level. However, they also argue that resistance jumps imply that the injection process cannot be described in terms of a second-order phase transition.

In our work, the conductance and resistance jumps arise as a finite-size effect, which can then be naturally interpreted in terms of conventional scaling arguments. To support this point of view, we have performed numerical simulations of these jumps for random resistor networks on the square lattice, and we have also constructed a simple-minded scaling argument for the jumps. Our scaling approach is based on the notion that the underlying source of the larger jumps is the breaking of singly connected bonds on various length scales. The predictions of this scaling argument are qualitatively consistent with our numerical results, suggesting that the basic mechanism for the conductance jumps is captured by a scaling approach. However, there are quantitative discrepancies with numerical results, and we discuss several possible reasons for this in the conclusions.

The simulations were performed on $L \times L$ square-lattice resistor networks, in which the opposite edges of the network were connected to bus bars, with one bus bar at potential $V = 0$ and the other at $V = L$. Periodic boundary conditions in the transverse direction are employed. The current flow problem was solved numerically by using the Fourier-accelerated conjugate gradient scheme, introduced by one of the authors [9]. We started with the completely occupied lattice, and bonds were removed, one by one, and at random. After each bond removal, the change in the conductance and resistance of the network was recorded, and figure 1 gives the results obtained from a typical single realisation. We observe many relatively small conductance jumps, but with a few much larger jumps interspersed. If the bond that is removed is singly

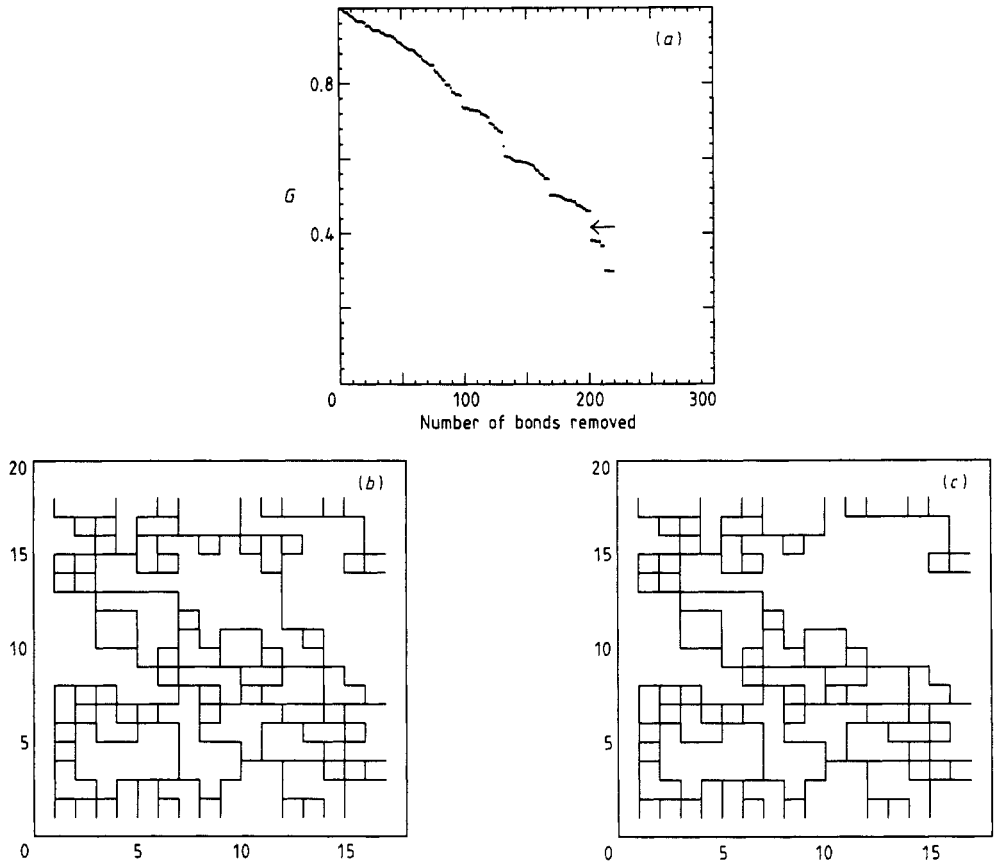

Figure 1. (*a*) Plot of the conductance of a 16×16 random resistor network as a function of the number of bonds removed from the system. This system has periodic boundary conditions in the transverse direction. We also show the configuration just before (*b*) and just after (*c*) the relatively large jump marked by the arrow in (*a*) has occurred.

connected on a relatively large scale, then there can also be a concomitant substantial change in the number of bonds in the backbone. This particular situation leads to a strong correlation between the large jumps in the conductance and relatively large changes in the structure of the conducting backbone, as illustrated in figure 1(*b*) and (*c*).

Clearly, if we were to average over many configurations, the conductance would become a smooth function. However, consider the change in the conductance of a particular configuration when a single bond is removed. As discussed above, this quantity becomes less predictable as $p \to p_c$. This behaviour is illustrated in figure 2, where we plot all the resistance jumps observed upon single bond removal, obtained from 12 configurations of a 32×32 network. Although the average size of the jumps is relatively small, the data also reveal that jumps on all scales exist, except when p is very nearly equal to 1. As in [8], we are then led to plot the number of conductance jumps larger than ΔG, $N_G(\Delta G)$ (where we henceforth consider only the absolute value of the conductance difference), and the number of resistance jumps larger than ΔR, $N_R(\Delta R)$, as shown in figure 3. For these plots, we obtained data for lattices of linear dimension $L = 4$ (averaged over 8000 configurations), 8 (200 configurations), 16 (60 configurations) and 32 (12 configurations).

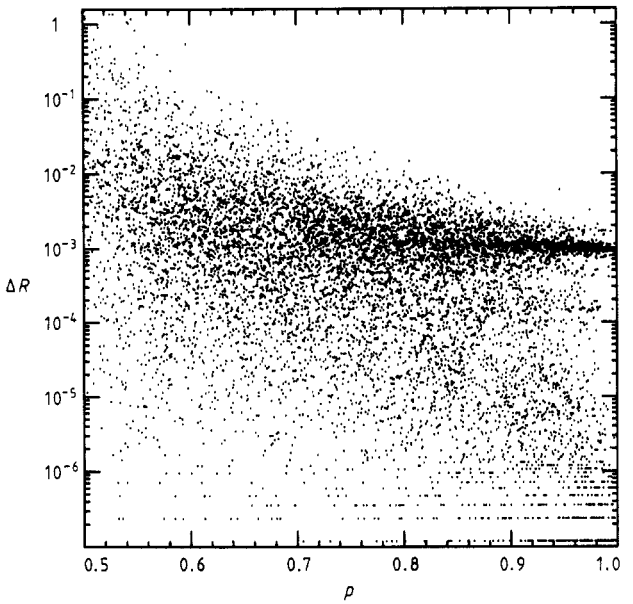

Figure 2. A scatter plot showing all of the resistance jumps observed from 12 configurations of a 32×32 network, when p is decreased from unity to p_c. For $p \simeq 1$, the resistance jumps are generally concentrated at one value, which corresponds to the behaviour expected from effective-medium theory. There also exist some very small jumps which correspond to deleting transverse bonds when $p \simeq 1$. However, for $p \simeq p_c$, resistance jumps on all scales exist. Note that the apparent discreteness of the smallest size jumps is due to the discreteness in the binning of the data.

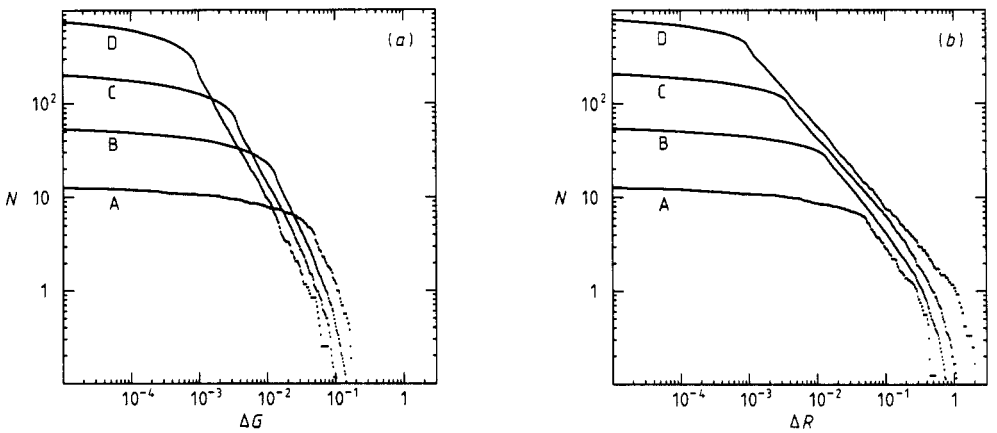

Figure 3. Double logarithmic plots of (a) the number of conductance jumps larger than ΔG, $N_G(\Delta G)$, against ΔG, and (b) the number of resistance jumps larger than ΔR, $N_R(\Delta R)$, against ΔR. Data for lattice sizes $L = 4$ (A), 8 (B), 16 (C) and 32 (D) are shown.

Both sets of data exhibit a rather well defined shoulder, below which the distribution of jump sizes is almost flat. The location of this shoulder corresponds closely to the value of the jump expected by removing a single vertical bond when $p \simeq 1$. Beyond this shoulder, the distributions of jump sizes appear to have a power law form, which extends over a substantial range for the larger lattice sizes. We therefore define two new exponents λ_G and λ_R by $N_G(\Delta G) \sim (\Delta G)^{-\lambda_G}$ and $N_R(\Delta R) \sim (\Delta R)^{-\lambda_R}$, respectively, to account for this power law behaviour. We estimated these exponents by performing linear least-squares fits to the data, when plotted on a double logarithmic scale, in which different portions of the data were removed in a systematic fashion. For $L = 4$ and 8, the fits were not very satisfactory owing to the relatively small range of apparent power law behaviour, but from the $L = 16$ and 32 data we estimate $\lambda_G \simeq 1.25$ and $\lambda_R \simeq 0.83$, with an uncertainty (statistical and systematic) of less than 10%.

Now we present our scaling argument for the behaviour of the conductance and resistance jumps. The predictions we obtain are essentially the same as those given by Roux and Wilkinson [7], although the details of our respective arguments differ. Our approach is based on a number of assumptions which are in the spirit of a mean-field theory. First, we use the nodes and links model [10] as our picture of a percolating system above the percolation threshold. We also assume that the conductance jumps are monotonically ordered in increasing size as the bond concentration is decreased towards the percolation threshold. Finally, we assume that the primary source of the conductance jumps is the breaking of singly connected bonds within one macrolink of the nodes and links model. When such a bond is broken, we then imagine that the system evolves to a new homogenous nodes and links picture in which the correlation length has changed from ξ to $\xi + d\xi$, rather than becoming a slightly distorted nodes and links picture with one macrolink removed.

More quantitatively, for a d-dimensional system of linear dimension L in which the correlation length is ξ, the conductance is given by $G \sim (L/\xi)^{d-2}\xi^{-\zeta/\nu}$, where ζ is the resistivity exponent. If a singly connected bond on length scale ξ is removed, then one of the links breaks. For this slightly defective lattice, we can appeal to effective-medium theory [1], from which it is immediate to show that the new value of the conductance, G', is related to G by $G[1-(\xi/L)^d]$. Therefore, the change in the conductance of the system is simply

$$\Delta G \sim (\xi/L)^2 \xi^{-\zeta/\nu}. \tag{1}$$

Furthermore, the number of singly connected bonds on length scale ξ in the system is given by $(L/\xi)^d \xi^{1/\nu}$ [6], so that to remove one singly connected bond on this scale requires a change in the absolute value of the bond concentration, Δp, proportional to $(\xi/L)^d \xi^{-1/\nu}$. Then as p is decreased from p to a value near p_c, the number of removals of singly connected bonds on length scale ξ varies as

$$N \sim \int_p^{p_c} \frac{dp}{\Delta p} \sim \left(\frac{L}{\xi}\right)^d. \tag{2}$$

In obtaining this last result, Δp was first rewritten as a function of $(p - p_c)$, and after the integral was performed, $(p - p_c)$ was then eliminated to arrive at an expression that depends only on ξ. Finally, by eliminating ξ between (1) and (2), we obtain the number of conductance jumps larger than size ΔG

$$N_G(\Delta G) \sim L^{-d\zeta/(2\nu-\zeta)} \Delta G^{-d\nu/(2\nu-\zeta)}. \tag{3}$$

Alternatively, we can use the exponent relation $t=(d-2)\nu+\zeta$ to rewrite the dependence of $N_G(\Delta G)$ on ΔG as $\Delta G^{-\lambda_G}$, with

$$\lambda_G=\frac{d\nu}{d\nu-t}. \tag{4}$$

The same line of reasoning leading to (1) can be applied to the behaviour of the resistance change in the system when one macrolink is broken. By this approach, we find

$$\Delta R\sim(\xi/L)^{2(d-1)}\xi^{\zeta/\nu}. \tag{1'}$$

Then by eliminating ξ in (1') and (2), we find the distribution of resistance jumps of size greater than or equal to ΔR given by

$$N_R(\Delta R)\sim L^{d\zeta/[2(d-1)\nu+\zeta]}\Delta R^{-d\nu/[2(d-1)\nu+\zeta]}. \tag{3'}$$

As in (3), we can rewrite the dependence of $N_R(\Delta R)$ on ΔR as $\Delta R^{-\lambda_R}$, with

$$\lambda_R=\frac{d\nu}{d\nu+t}. \tag{4'}$$

Using the currently accepted values for the exponents t and ν, in two dimensions, we obtain $\lambda_G\simeq 1.92$ and $\lambda_R\simeq 0.676$. Furthermore, in two dimensions, the L dependence of the jump distribution is given by $L^{-1.84}$ for the conductance and $L^{+0.648}$ for the resistance. By comparing with the data of figure 3, we find qualitative but not quantitative agreement. Interestingly, the numerical results of Roux and Wilkinson [7] for the resistance jumps do agree extremely well with the prediction given in equation (4'). However, their data were based on a single realisation of a 20^3 network of the simple cubic lattice with both substitutional (percolation) disorder and disorder in the value of the bond conductances. Therefore, it is not clear to what degree (if any) the disorder in the bond conductances and the potential statistical errors affect quantitative results for the jump distribution. It would be interesting to test the scaling prediction more conclusively in three dimensions by considering the conductance jumps, and also by studying the dependence of the resistance and conductance distributions on lattice size for purely percolation disorder.

However, on general grounds, one may expect that a scaling argument based on the mean-field nodes and links picture will give more accurate results in three dimensions than in two. Within the framework of a nodes and links picture, it is to be expected that inhomogeneities in the properties of the macrolinks will be more pronounced in lower dimensions, and this is not accounted for in our scaling approach. Furthermore, by considering deterministic models of the percolating backbone, such as the 'bubble' model [11], we can gain additional insights about why scaling may not fully account for the distribution of conductance jumps. In the bubble model, the largest conductance jump is found to occur when a doubly connected bond pair becomes a singly connected bond. When a singly connected bond exists, the network conductance coincides with the voltage drop through the singly connected bond, and this latter quantity has been shown to grow logarithmically with the linear dimension of the system [11-13]. As a result, the corresponding conductance jump should also reflect this logarithmic size dependence, and this may partially account for the discrepancies between our numerical results and the scaling predictions.

In summary, we have studied the behaviour of the conductance and resistance jumps when bonds are removed, one by one and at random, from a resistor network.

These jumps are a manifestation of a finite-size effect in which the removal of a single bond is equivalent to a finite change in the bond concentration. The size distribution of conductance and resistance jumps appears to have a power law dependence on size, and a very approximate scaling theory for the exponents characterising these distributions has been given. While we have focused only on the behaviour of the distribution of jump sizes, we believe that there are other useful applications for the conductance and resistance jumps. In particular, we believe that it may be possible to obtain accurate estimates for the conductivity exponent t from the concentration dependence of a single configuration of a random resistor network by exploiting information about the conductance jumps in an optimal way. Work along these lines is in progress.

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